

Transition redshift in $f(T)$ cosmology and observational constraints

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(Dated: June 19, 2015)

We extract constraints on the transition redshift z_{tr} , determining the onset of cosmic acceleration, predicted by an effective cosmographic construction, in the framework of $f(T)$ gravity. In particular, employing cosmography we obtain bounds on the viable $f(T)$ forms and their derivatives. Since this procedure is model independent, as long as the scalar curvature is fixed, we are able to determine intervals for z_{tr} . In this way we guarantee that the Solar-System constraints are preserved and moreover we extract bounds on the transition time and the free parameters of the scenario. We find that the transition redshifts predicted by $f(T)$ cosmology, although compatible with the standard Λ CDM predictions, are slightly smaller. Finally, in order to obtain observational constraints on $f(T)$ cosmology, we perform a Monte Carlo fitting using supernova data, involving the most recent union 2.1 data set.

PACS numbers: 04.50.Kd, 98.80.-k, 95.36.+x

I. INTRODUCTION

Observational evidences imply that the universe is undergoing a phase of anomalous acceleration after a precise time, usually named the *transition time* [1, 2]. In particular, the corresponding transition redshift, z_{tr} , indicates at which stage the universe changed its dynamical properties and started accelerating after a phase of deceleration [3]. Recently, it has been argued that constraining z_{tr} provides information on the form of the fluid responsible for the observed universe speeding up. Consequently, z_{tr} may likely reveal possible new gravitational physics due to modifications of Einstein's gravity [4].

The fluid which triggers the current universe acceleration is often referred to as *dark energy* and fills more than the 70% of the whole universe energy budget [5]. The standard cosmological model assumes that the dark energy source is supplied by the existence of a non-zero cosmological constant Λ . The corresponding paradigm, named the Λ CDM model, is constructed by employing a net matter density composed by baryons and cold dark matter, with a constant dark energy term $\Omega_\Lambda \equiv \frac{3\Lambda}{8\pi G}$ [6]. Even though the model likely represents the simplest approach for describing universe's dynamics, amongst others the cosmological constant does not furnish an explanation to the *coincidence problem* between matter and dark energy magnitudes. In other words, since the cosmological constant does not evolve in time, it is improbable that the ratio between matter and dark energy densities is so close today [7]. Additionally, quantum field theory predictions forecast an enormous value for the cosmological constant if compared with the one measured by current cosmological observations. This issue is the well known *fine-tuning* problem and represents a challenge to understand the physical origin of the cosmological con-

stant itself [8]. Due to the above caveats, one can modify the universe content, and attribute the dark energy sector to a canonical scalar field, a phantom field, to the combination of both fields in a unified model, or proceed to more complicated constructions (for reviews see [2, 9]).

An alternative way to reproduce the universe dynamics is by extensions of general relativity by means of additional degrees of freedom, which do not violate the equivalence principle, and represent a bid to formulate a semi-classical scheme for both late and early-time universe [10]. In the usual approach to modify gravity, one starts by the usual curvature formulation of general relativity, and replaces the Ricci scalar R in the Einstein-Hilbert action by arbitrary functions of it, or even more complicated curvature invariants. However, alternatively one can use as a base the torsional formulation of general relativity, namely the so called "teleparallel equivalent of general relativity" [11], and modify its action instead. In particular, in teleparallel gravity [12] the gravitational field is described not by the curvature tensor but by the torsion one, and thus the corresponding Lagrangian, namely the torsion scalar T , is constructed by contraction of the torsion tensor in a similar way that in usual general relativity the Lagrangian, namely the curvature scalar R , is constructed by contractions of the curvature tensor. Hence, similarly to the $f(R)$ extension of general relativity, one can construct the $f(T)$ extension of teleparallel equivalent of general relativity [13, 14]. The interesting feature is that although general relativity coincides completely with teleparallel equivalent of general relativity, $f(T)$ gravity is different from $f(R)$ one, thus it is a novel gravitational modification with rich cosmological implications [14–16].

With those considerations in mind, in this work we are interested in describing the dark energy effects, di-

rectly calculating the corresponding transition redshift z_{tr} that is predicted in $f(T)$ cosmology. In order to do so, we only consider those $f(T)$ models which are consistent with present-time cosmographic constraints. Hence, we aim to obtain cosmographic bounds on the $f(T)$ scenarios, by considering the modified Friedmann equations, and then get the corresponding limits on z_{tr} . The main advantage of using cosmography is that the value of z_{tr} is reconstructed by means of a model-independent procedure. Rephrasing it differently, we are able to distinguish which classes of $f(T)$ gravity pass the cosmographic requirements and thus are viable, by inferring the limits over z_{tr} that those classes predict. In particular, we compare this transition epoch with the one determined by the standard cosmological paradigm, and we propose an effective cosmological model capable of reproducing the cosmographic constraints and compatible with the limits on z_{tr} . Additionally, to enable our treatment, we consider the use of the luminosity distance and we match cosmic union 2.1 supernova data [17] with the cosmographic expansions. Thus, we evaluate the corresponding deceleration parameter and we show that the transition redshift is effectively comparable to the one predicted by the Λ CDM approach.

As we will see, the limits on z_{tr} show that the considered $f(T)$ classes reduce to the Λ CDM model in the lowest redshift domain, in agreement with [18, 19]. This feature indicates that the role played by the cosmological constant may be reinterpreted as a limiting case of a more general extension, and thus from those cosmographic corrections we show that small discrepancies occur at $z \leq 1$, whereas higher departures might be expected at high-redshift regimes. At this point, we involve a Monte Carlo fitting procedure based on the Metropolis algorithm, in order to compare our effective cosmological model with present-time data. Numerical limits, priors and final outcomes, testify the efficiency of our approach, showing that the effective torsional dark energy naturally satisfies the cosmographic requirements, and hence it may be a candidate as a valid alternative to describe the universe dynamics.

The paper is organized as follows. In Sec. II we describe the techniques for recovering the cosmographic settings on the $f(T)$ classes of models and we moreover propose how to obtain $f(T)$ reconstructions. In Sec. III we enumerate the properties of the transition redshift and its important role in modern cosmology. Furthermore, we describe the main consequences in $f(T)$ gravity and we show how the modified Friedmann equations changed when the transition occurred. In Sec. IV we summarize the cosmographic results and we propose an effective reconstruction of $f(T)$ cosmology. To do so, we infer the deceleration parameter for the effective torsional dark energy models and finally we show the numerical priors on the transition redshift z_{tr} predicted by our paradigm. In Sec. V we compare the cosmological consequences of the examined models with modern data, employing the use of the union 2.1 supernova survey. We determine the free

parameters of our approach and we show that the dark energy corrections are compatible with the bounds offered by alternative dark energy models. Finally, in Sec. VI we summarize the conclusions and perspectives of our approach.

II. THE PROCEDURE FOR $f(T)$ RECONSTRUCTION FROM COSMOGRAPHY

In teleparallel formulation of gravity, as well as in its $f(T)$ extension, one uses the vierbein fields e_A^μ , which form an orthonormal base for the tangent space at each point x^μ defined on a generic manifold, and thus the metric reads as $g_{\mu\nu} = \eta_{AB} e_A^\mu e_B^\nu$ (in the following greek indices and Latin indices span the coordinate and tangent spaces respectively). Additionally, instead of the torsionless Levi-Civita connection one uses the curvatureless Weitzenböck one $\overset{\text{w}\lambda}{\Gamma}_{\nu\mu}^\lambda \equiv e_A^\lambda \partial_\mu e_\nu^A$ [12], and therefore the gravitational field is encoded in the torsion tensor

$$T^\rho{}_{\mu\nu} \equiv e_A^\rho (\partial_\mu e_\nu^A - \partial_\nu e_\mu^A). \quad (1)$$

Hence, the Lagrangian of teleparallel gravity, namely the torsion scalar T , is constructed by contractions of the torsion tensor as [12]

$$T \equiv \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T^{\rho\mu\nu} T_{\nu\mu\rho} - T_{\rho\mu}{}^\rho T^{\nu\mu}{}_\nu. \quad (2)$$

Finally, one can extend teleparallel gravity and construct the action of $f(T)$ gravity as [13, 14]

$$\mathcal{S} = \int d^4x e \left[\frac{f(T)}{2\kappa^2} \right], \quad (3)$$

where $e = \det(e_A^\mu) = \sqrt{-g}$ and κ^2 is the gravitational constant.

The general field equations of $f(T)$ gravity are obtained by varying the action $\mathcal{S} + \mathcal{S}_m$, with \mathcal{S}_m the matter action, in terms of the vierbeins, and they read as

$$e^{-1} \partial_\mu (e e_A^\rho S_\rho{}^{\mu\nu}) f' + e_A^\rho S_\rho{}^{\mu\nu} \partial_\mu (T) f'' - f' e_A^\lambda T^\rho{}_{\mu\lambda} S_\rho{}^{\nu\mu} + \frac{1}{4} e_A^\nu f = \frac{\kappa^2}{2} e_A^\rho T^{(m)\nu}{}_\rho, \quad (4)$$

where the tensor $S_\rho{}^{\mu\nu} \doteq \frac{1}{2} (K^{\mu\nu}{}_\rho + \delta_\rho^\mu T^{\alpha\nu}{}_\alpha - \delta_\rho^\nu T^{\alpha\mu}{}_\alpha)$ is defined in terms of the co-torsion $K^{\mu\nu}{}_\rho \doteq -\frac{1}{2} (T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu})$, and where $T^{(m)\nu}{}_\rho$ is the energy-momentum tensor corresponding to \mathcal{S}_m . In (4) the primes denote derivatives with respect to T . Finally, since for $f(T) = T$ equations (4) provide exactly the same equations with general relativity, that is why the theory with $f(T) = T$ was named by Einstein “teleparallel equivalent of general relativity” [11].

In order to apply $f(T)$ gravity in a cosmological framework we assume a spatially-flat Friedmann-Robertson-Walker metric $ds^2 = dt^2 - a(t)^2 (dr^2 + r^2 d\Omega^2)$, with $d\Omega^2 \doteq d\theta^2 + \sin^2 \theta d\phi^2$, which can arise from the vierbein

$e_\mu^A = \text{diag}(1, a, a, a)$. In this case, the field equations (4) give rise to the modified Friedmann equation

$$H^2 = \frac{1}{3}(\rho_m + \rho_T), \quad (5a)$$

$$\dot{H} = -\frac{1}{2}(\rho_m + p_m + \rho_T + P_T), \quad (5b)$$

with $H \doteq \frac{\dot{a}}{a}$ the Hubble parameter and dots indicating derivatives with respect to the cosmic time. In the above expressions ρ_m and p_m are the energy density and pressure of the matter sector considered to correspond to a perfect fluid, and moreover from now on we use units in which $\kappa^2 = 1$. Furthermore, we have introduced the energy density and pressure of the effective dark energy sector, which incorporates the torsional modifications, as

$$\rho_T \doteq -\frac{f}{2} - \frac{T}{2} + T f', \quad (6a)$$

$$P_T \doteq \frac{1}{2} \left[\frac{f - f' T + 2T^2 f''}{f + 2T f''} \right]. \quad (6b)$$

Thus, the dark energy equation-of-state parameter writes as $w_{DE} \doteq w_T = P_T/\rho_T$. Finally, note that for the FRW geometry, the calculation of the torsion scalar (2) leads to the useful relation

$$T = -6H^2. \quad (7)$$

The issue of finding out a form for the dark energy equation of state passes through the determination of the most viable forms of $f(T)$. If one knows the $f(T)$ form, it is possible to infer the interpretations of ρ_T and P_T as terms associated to torsional dark energy, i.e. a torsional contribution driving the observed cosmic acceleration. The idea to get a viable $f(T)$ function lies on requiring that at small redshift the $f(T)$ model reproduces the observational data and predicts a compatible transition redshift. Hence, the strategy of this manuscript is to frame a phenomenological reconstruction of $f(T)$ and its derivatives in terms of cosmography, which becomes a sort of initial settings for $f(T)$ models. Having in mind these cosmographic requirements, we simply impose the validity of the cosmological principle [20], the geometrical setting of the scalar curvature [21], and the possibility of expanding $f(T)$ and its derivatives around present time in Taylor series [22]. We discuss below each of those three requirements, in order to define the cosmographic series and its application in $f(T)$ cosmology.

- First, employing the cosmological principle permits to frame the universe expansion history in terms of a single parameter, namely the scale factor $a(t)$ which enters the Friedmann-Robertson-Walker metric as function of the cosmic time only. One gathers viable outcomes imposing that this function may be expanded in Taylor series around present time, and constraining the corresponding Taylor coefficients associated to the scale-factor

derivatives. This strategy compares $a(t)$'s derivatives *directly* with cosmic data and may be used as a reconstruction for the $a(t)$ shape. This benefit allows one to distinguish among all paradigms, derived from imposing the form of $f(T)$, the ones whose cosmographic requirements better match with data.

- The second caveat is the issue of spatial curvature which leads to a degeneracy problem between its value and the variation of the acceleration. It has been proved that photon geodesics change their paths according to its value. Thus, expanding a physical quantity into a cosmographic series needs to fix somehow the value of spatial curvature, in order to allow cosmography to be as model-independent as possible [23]. According to previous approaches [24, 25], one imposes geometrical bounds on Ω_k by assuming the matching between early and late time observations [26]. We therefore assume that the universe is spatially flat, with possible small deviations which do not influence the whole dynamics.
- Finally, since all observable quantities of interest are assumed to smoothly evolve as the universe expands, it is licit to assume that Taylor expansions may be easily accounted and no saddle points or poles occur. It follows that all functions are analytic and the cosmographic treatment is perfectly plausible [27, 28]. After those properties, one soon expands in Taylor series the scale factor $a(t)$ as

$$a(t) \doteq \sum_{l=0}^{\infty} \frac{1}{l!} a_p \bar{t}^l \approx a_0 + a_1 \bar{t} + a_2 \bar{t}^2 + \dots, \quad (8)$$

where $a_p \doteq \frac{d^p a}{dt^p}$, with $\bar{t} \doteq t - t_0$ and t_0 the present time. Finally, it proves convenient to express the observable quantities under interest in terms of the redshift $z = -1 + a_0/a$.

Amongst all observables, we are much interested in the use of the luminosity distance, since we will use supernovae Ia type to fix our cosmological bounds. Hence, imposing $a_0 = 1$ in (8) and inserting it in the definition of the luminosity distance:

$$D_L = (1+z) \int_0^z \frac{d\xi}{H(\xi)}, \quad (9)$$

we write down the Taylor series around $z = 0$ as

$$\begin{cases} D_L = \frac{z}{H_0} \tilde{d}_L(z; \theta), \\ \tilde{d}_L(z; \theta) = 1 + d_{L1}z + d_{L2}z^2 + \dots, \end{cases} \quad (10)$$

truncated at the third order. Moreover, applying also the definitions

$$\dot{H} = -H^2(1+q), \quad (11a)$$

$$\ddot{H} = H^3(j + 3q + 2), \quad (11b)$$

we obtain the corresponding coefficients of the Taylor expansion in terms of the cosmographic series:

$$d_{L1} = \frac{1 - q_0}{2}, \quad (12a)$$

$$d_{L2} = \frac{1 - q_0(1 + 3q_0) + j_0}{6}. \quad (12b)$$

In these expressions we have introduced the deceleration and jerk parameters as

$$q \doteq -\frac{1}{aH^2} \frac{d^2 a}{dt^2}, \quad (13a)$$

$$j \doteq \frac{1}{aH^3} \frac{d^3 a}{dt^3}, \quad (13b)$$

indicating respectively whether the universe is accelerating or not and how the acceleration changed sign, with the subscript “0” denoting the value of a quantity at present. Observations indicate $j_0 > 0$ and then testify that the transition time occurred. However, there still exists a tension between the possibilities $0 < j_0 < 1$ and $j_0 > 1$ [29].

For the sake of completeness, we notice that it would be easy to arbitrarily extend the cosmographic series up to higher orders. For example, the next term, entering the Taylor expansion of the luminosity distance, i.e. the fourth order, would linearly depend on the snap $s \equiv \frac{1}{aH^4} \frac{d^4 a}{dt^4}$ evaluated at present time¹. However, expanding beyond the third-order leads to a non-clear physical interpretation of the corresponding coefficients. In other words, using extended cosmographic series at higher order could become quite non-predictive for our analysis, since it is difficult to physically bound coefficients beyond j_0 . Indeed, in order to improve constraints over cosmographic coefficients at arbitrary orders, one can imagine to adopt either experimental combined tests or different definitions of cosmic distances [21]. In particular, using combined tests may generally reduce the numerical intervals of cosmographic coefficients, while any alternative distance definition would be plagued by inaccurate under/over-estimations of the cosmographic parameters, typically due to the limited cosmic available data. In addition, present time combined tests may only provide tighter ellipses where the cosmographic coefficients span, albeit definitive constraints up to 5σ confidence level would be hardly determined.

A straightforward example of the difficulty to bound the cosmographic parameters is offered by supposing to fix the today spatial curvature with arbitrary accuracy. In that case, it would be possible to better circumscribe the parameters beyond the third order (including jerk, snap, or higher terms)². Unfortunately, from this proce-

dure we can only partially exclude regions where those intervals run, getting corresponding upper and lower limits which characterize each coefficients of the Taylor expansions³. Recent analysis fixes the following limits over the cosmographic coefficients [20]:

$$q_0 \in [-0.9; -0.4], \quad (14a)$$

$$j_0 \in [0.8; 2], \quad (14b)$$

$$s_0 \in [-1; 7], \quad (14c)$$

all evaluated at a 2σ confidence level. It is immediately clear that badly constrained results can be obtained going beyond the third order of the cosmographic expansion. In particular, as stressed above, the issue associated to those regions is that they correspond to narrow (and often long) ellipses which are characterized by high error dispersions. Thus, even modern data seem to focus on strict regions for H_0, q_0 and j_0 , whereas do not account for the sign of s_0 in the same way.

In addition, as shown in [30], one may relate cosmography to the cosmic equation of state. From this fact, it is easy to show that the second order derivative of the cosmic pressure with respect to the total density linearly depends upon s_0 itself. Alternatively, even the second derivative of the acceleration parameter, i.e. $\frac{d^2 q}{dz^2}$, cannot be accurately constrained with current numerics on s_0 reported in (14).

Summarizing, adopting present data, all cosmographic quantities suffer from badly bounded numerical results, so that the use of s_0 in our approach would influence the experimental analysis itself, providing broadening systematics and higher dispersions in the evaluations of the transition redshift z_t . For those reasons, in order to consider fourth order expansions, one should adopt improved intervals of cosmic data or using distance definitions that do not somehow depend on scalar curvature, i.e. where the scalar curvature is not fixed as a prior coming from other observations. Hence, in this paper, we definitively use a truncated third order Taylor expansions of D_L thanks to the above considerations. In so doing, the numerical results of H_0, q_0 and j_0 lead to acceptable dispersions and allow us to better circumscribe the intervals for z_{tr} in a more suitable way.

Thus, from the above analysis it follows that the cosmographic series is the set of coefficients evaluated at present time. The cosmographic series has been built up in function of a Taylor series expanded around $z = 0$, albeit it is possible to handle it also in terms of the cosmic time. This is clearly possible involving the definition of the redshift in terms of the cosmic time as:

$$\frac{dz}{(1+z)} = -H(z)dt. \quad (15)$$

¹ In general, it is straightforward to prove that any order linearly depends on the cosmographic n -term [30].

² For our purposes, we follow the strategy to take a vanishing scalar curvature to show that, under this hypothesis, the cosmographic coefficients allow compatible transition redshifts in $f(T)$ gravity.

³ Possible examples of those regions are given in [31]. Here, the authors showed that by means of combined tests, with older supernova data sets, the jerk parameter should be positive at the 92% confidence level.

Since the cosmographic series may be expressed in terms of the scale-factor derivatives, it does not depend upon the particular choice of the cosmological model. This property represents a key to conclude that imposing a cosmological model *a priori* is unnecessary and any modified gravity may be limited by assuming the cosmographic requirements. We here follow the technique of reconstructing the $f(T)$ models by means of late-time cosmography. To do so, we rewrite the luminosity distance (9) in terms of $f(T)$ derivatives. This is possible since there exists a direct correspondence between q_0 and j_0 with the $f(T)$ form and its derivatives. In particular, rewriting (12) as function of $f(T)$, we frame the effective model derived from the torsional dark energy by directly comparing D_L with data. Rephrasing it differently, instead of using q, j, \dots we consider $f(T), f'(T), f''(T), \dots$ and we obtain the numerical outcomes on those quantities. The cosmographic constraints on $f(T)$ and its derivatives, point out the numerical priors that we use as *initial settings* for reconstructing the shape of viable effective $f(T)$ models, which reproduce dark energy at small redshift. Afterwards, we predict the transition redshift from our effective model and we understand whether our model indicates a viable z_{tr} if compared with the Λ CDM predictions. We will report the connections between the cosmographic series and the $f(T)$ derivatives in Sec. IV.

III. THE $f(T)$ TRANSITION REDSHIFT

In the scenario at hand, the $f(T)$ term drives the dark energy contribution, interpreting the dark sector as due to a *torsional dark energy*. It is widely believed that the dark energy contribution dominates over matter at our time, while it appears negligible at higher redshift regimes. The type of the transition and the time at which it occurs are extremely relevant, since they indicate the dark energy nature and may also provide information on how the dark energy evolves in time. In particular, the transition time and correspondingly the transition redshift emphasize the change from decelerated to accelerated cosmological expansion, and represent a prediction of any particular model involved to describe the universe expansion history. In other words, direct measurements of the transition redshift provide direct information on both the deceleration and acceleration epochs.

To show how to investigate z_{tr} in the framework of $f(T)$ gravity, let us consider the definition of the transition redshift, which occurs at a zero of the deceleration parameter q . We will find out the transition redshift z_{tr} for a class of cosmographic $f(T)$ models, and we will also compare it with standard model predictions and with recent bounds on z_{tr} itself.

Passing through the phase of transition between matter and dark energy dominance, and assuming the matter to be dust (i.e. $P_m = 0$), it is useful to combine the two Friedmann equations (5a), (5b) to obtain the torsional

pressure in terms of the deceleration parameter as:

$$P_T = H^2(2q - 1), \quad (16)$$

where we made use of relation (13a). At the transition time we therefore obtain the value of the torsional pressure as

$$P_T(z_{tr}) = -H_{tr}^2, \quad (17)$$

which corresponds to $q = 0$ at the transition redshift z_{tr} , with Hubble rate H_{tr} . This expression is equivalent to the standard barotropic dark-energy pressure in the framework of general relativity given by [32]:

$$P_{tr} = -H_{tr}^2. \quad (18)$$

In particular, the two results, (17) and (18), lead to the same formal outcome. In fact, assuming H_{tr} to be positive definite, both the torsional and standard dark-energy pressures are negative at the transition. However, the physical meaning behind (17) and (18) is different, in the sense that in the first case the transition is induced by the torsional terms, while in the standard approach the transition is realized due to the dark energy or curvature terms.

In the standard Λ CDM cosmological model, the cosmological constant contributes about 70% of the present cosmological energy budget and the consequence on cosmology lies on an evolving deceleration parameter q of the form [1]

$$q_\Lambda = -1 + \frac{3\Omega_{m,0}(1+z)^3}{2 + 2\Omega_{m,0}z[3 + z(3+z)]}, \quad (19)$$

with $\Omega_{m,0}$ the present matter density parameter, and where the corresponding transition redshift formally is given by

$$z_{tr} = \left(\frac{1}{H} \frac{dH}{dz} \right)^{-1} \Big|_{z=z_{tr}} - 1, \quad (20)$$

which has been obtained assuming $\ddot{a} = 0$. Thus, the Λ CDM model gives an exact solution for z_{tr} , of the form

$$z_{tr,\Lambda} = \left[2 \frac{(1 - \Omega_{m,0})}{\Omega_{m,0}} \right]^{1/3} - 1. \quad (21)$$

In the following section we will use relations (16) and (17) in order to infer the numerical values of the torsional pressure at the transition time. Afterwards, we will quantify the difference of the standard predictions of the Λ CDM model with those obtained in the present cosmographic approach. We will therefore predict z_{tr} for $f(T)$ cosmology and we will compare it with expression (21).

IV. RECONSTRUCTING EFFECTIVE COSMOGRAPHIC $f(T)$ MODELS

Let us now apply the approach described in the previous two sections and proceed to the reconstruction of

effective cosmographic $f(T)$ models. We start by using (7), as well as (13a),(13b), in order to calculate the time derivatives of T as

$$\dot{T} = 12 H^3 (1 + q), \quad (22a)$$

$$\ddot{T} = -12 H^4 (q^2 + j + 5q + 3), \quad (22b)$$

with the connection between the torsion and P_T reading as

$$T = \frac{6P_T}{1 - 2q}. \quad (23)$$

Hence, at the transition time, we have

$$\dot{T} = 12 H_{tr}^3, \quad (24)$$

$$\ddot{T} = -12 H_{tr}^4 (j_{tr} + 3). \quad (25)$$

The cosmographic requirements give the connection between f , f' and f'' from the modified Friedmann equations, namely [22]

$$f(T_0) = 6H_0^2 (\Omega_{m,0} - 2), \quad (26a)$$

$$f''(T_0) = \frac{1}{6H_0^2} \left[\frac{1}{2} - \frac{3\Omega_{m,0}}{4(1+q_0)} \right], \quad (26b)$$

and moreover $f'(T_0) = 1$ to guarantee the Solar-System constraints in order to preserve the value of G at our time [22].

Thus, using expressions (22) and (26), we acquire the priors on T , $f(T)$ and $f''(T)$ as

$$f(T_0) \in [-5.23; -4.79], \quad (27a)$$

$$f''(T_0) \in [-0.19; 0.01], \quad (27b)$$

$$T_0 \in [-3.11; -2.77], \quad (27c)$$

$$\dot{T} \in [0.75; 2.24], \quad (27d)$$

$$\ddot{T} \in [-7.26; -0.36], \quad (27e)$$

which have been obtained assuming a normalized Hubble rate $H_0 \in [0.68; 0.72]$ and a mass density $\Omega_{m,0} \in [0.274; 0.318]$, with $q_0 \in [-0.8, -0.5]$, $j_0 \in [0.5, 1.5]$ and $j_{tr} \approx j_0$ [33]. The last condition has been imposed assuming that the universe is slightly evolving in the redshift domain $z \leq 1$. We stress that the priors (27) are the requirements that determine whether a specific $f(T)$ form is viable or not.

The strategy is the following: we assume the validity of the cosmographic series as the initial conditions of the modified Friedmann equations, and then we integrate the first Friedmann equation. Thus, we infer the numerical values of $H(z)$ for different redshifts, and we separately extrapolate those points, determining a list of numbers for $H(z)$ and z . Finally, through the use of testing functions, we reconstruct an effective $f(T)$ which reproduces the numerical limits. Hence, from this function one obtains a parameterized cosmological model, which departs from the Λ CDM scenario, corresponding to a varying dark-energy sector.

Our treatment suggests that a possible approximation of the dark-energy density term ρ_{DE} may be

$$\rho_{DE} \approx \log \left[\alpha + \beta \sum_{i=0}^{\mathcal{N}} a^i \right]. \quad (28)$$

Truncating at the second order in a , we obtain the Hubble rate as

$$\frac{H^2}{H_0^2} = \Omega_m(z) + \log[\alpha + \beta(2 - 3a + a^2)], \quad (29)$$

where we considered $\Omega_m \equiv \Omega_{m,0}(1+z)^3$. The parameter α is fixed in order to guarantee that at $z = 0$ the Hubble rate is identically $H = H_0$. Therefore, we have

$$\alpha = e^{1 - \Omega_{m,0}}. \quad (30)$$

Hence, the cosmographic reconstruction of torsional dark energy provides a deceleration parameter of the form

$$q = \frac{3\Omega_{m,0}(1+z)^5\alpha + \beta + 3z[1 + \Omega_m(z)(1+2z)]\beta}{2[(1+z)^2\alpha + z(1+2z)\beta] \left[\Omega_m(z) + \log \left(\frac{\alpha + \beta z(1+2z)}{(1+z)^2} \right) \right]} - 1. \quad (31)$$

In Fig. 1 we depict the behaviors of $H(z)/H_0$ and $q(z)$, given in (29) and (31) respectively. We deduce that up to the redshift domain $z \leq 2$, our approach is compatible with the standard cosmological model, and in fact only small differences occur between our predictions and the Λ CDM ones, which are slightly larger. This is due to the fact that our $H(z)$ parameter indicates a dark-energy evolution which does not departure significantly from the case of a constant dark-energy term at small redshifts. Hence, our Hubble rate well approximates the standard Λ CDM contribution, slightly evolving as the redshift increases. This is more evident in Fig. 2, in which we plot the dark-energy term (28), normalized by means of the standard critical density $\rho_c \equiv \frac{3H_0^2}{8\pi G}$.

Afterwards, linearizing the deceleration parameter around $z = 0$, keeping first-order terms, we find the transition redshift as

$$z_{tr} \simeq \frac{\alpha^2(\Omega_{m,0} + \log \alpha)(2 \log \alpha - \Omega_{m,0} - \beta)}{[9\Omega_{m,0}\alpha^2 + (\alpha - \beta)\beta] \log \alpha - \beta[\beta + \Omega_{m,0}(5\alpha + \beta)]}. \quad (32)$$

We mention that this approximation is efficient, since one expects a transition at $z \leq 1$ and therefore the linearized q does not substantially differ from the exact value.

Having in mind the form of the Hubble rate, we can infer limits over $\Omega_{m,0}$ and β . This permits one to determine z_{tr} from expression (32). In the next section we describe the fitting procedure using supernova data, and we extract numerical bounds on the free parameters of our cosmographic torsional dark-energy scenario.

V. THE MATCHING WITH OBSERVATIONS

The above approach provided a particular set of cosmographic quantities related to the $f(T)$ form. Corre-

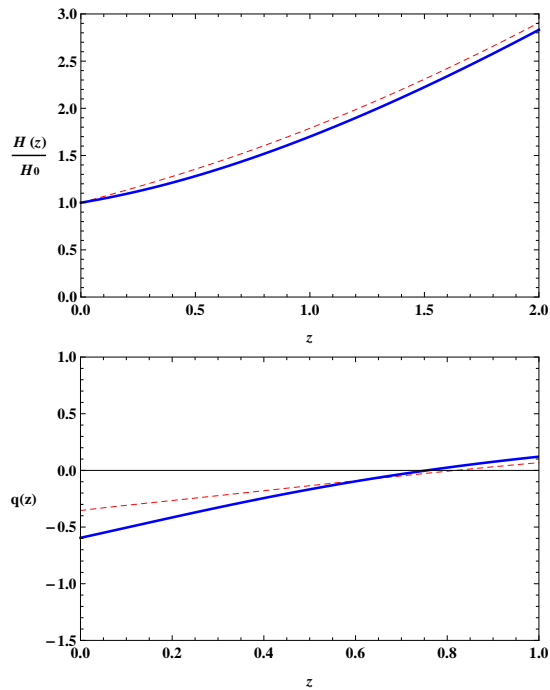


Figure 1: The evolution of $H(z)/H_0$ (upper graph) and $q(z)$ (lower graph), according to viable $f(T)$ cosmology (blue-solid curves) versus the Λ CDM predictions (red-dashed curves). We employed the indicative values $\Omega_{m,0} = 0.27$ and $\beta = 1$ and we normalized through $H_0 = 100 \text{ Km Mpc}^{-1} \text{ s}$.

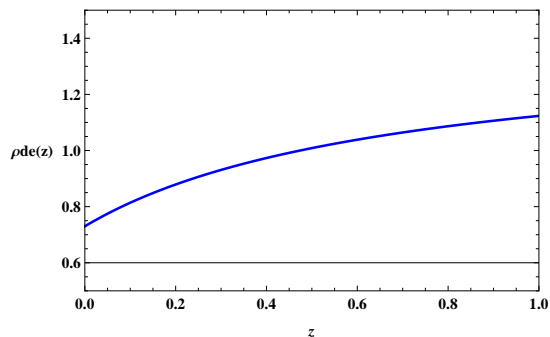


Figure 2: The evolution of the dark-energy density according to viable $f(T)$ cosmology (blue-solid curves) versus the Λ CDM predictions (red-dashed curves). The two cosmological paradigms exhibit very similar behaviors, and thus the Λ CDM curve is almost indistinguishable from the viable $f(T)$ one. We employed the indicative values $\Omega_{m,0} = 0.27$ and $\beta = 1$ and we normalized through $H_0 = 100 \text{ Km Mpc}^{-1} \text{ s}$.

spondingly, the effective Hubble rate was built in terms of corrections to the simple teleparallel gravity, that is to general relativity. Hence, these corrections are due to the difference between $f(T)$ cosmology and the standard paradigm of Λ CDM cosmology. These terms may be compared with the cosmological constant value without showing great departures at small redshift regimes, as we discussed in the above section. The contribution due to the $f(T)$ sector needs to match an adequate conver-

gence at small and high redshift domains. Hence, it is required to obtain bounds on the observational parameters, in order to understand whether the cosmological model at hand passes or not the observational constraints.

In order to proceed, we perform the analysis by involving the Monte Carlo technique with the use of the union 2.1 supernova compilation [17]. This survey is built up by 580 measurements of apparent magnitudes, with the corresponding redshifts and magnitude errors. Assuming a Gaussian distribution, one acquires the relevant fact that the luminosity distance may be rewritten in terms of the cosmographic series itself. Thus, all observations may be performed by directly fitting D_L with union 2.1 data.

We employ type Ia supernova observations since they probably represent the most suitable cosmic compilation. The role of supernovae has been crucial for cosmological parameter-fittings, since supernovae are considered standard candles. It follows that their luminosity curves are easily related to distances themselves⁴.

The union 2.1 data set is capable of reducing previous systematics, entered in old catalogs, for instance in union and union 2 [35, 36]. Hence, it is easy to show that one can use the well known χ -squared function, which is commonly involved to quantify theoretical and observational distance modulus. In particular, one defines it as [37]

$$\chi^2 = \mathcal{A} - \frac{\mathcal{B}^2}{\mathcal{C}} + \mathcal{D}, \quad (33)$$

where

$$\begin{aligned} \mathcal{A} &= \mathbf{x}^T C^{-1} \mathbf{x}, \\ \mathcal{B} &= \sum_i (C^{-1} \mathbf{x})_i, \\ \mathcal{C} &= \text{Tr}[C^{-1}], \\ \mathcal{D} &= \log \left(\frac{\mathcal{C}}{2\pi} \right), \end{aligned} \quad (34)$$

with C the covariance matrix of observational data, and where \mathbf{x} is

$$\mathbf{x}_i = 5 \log_{10} \left[\frac{D_L(z_i; \theta)}{\text{Mpc}} \right] + 25 - \mu_{obs}(z_i). \quad (35)$$

In particular, having a spatially flat universe, the luminosity distance simply reduces to

$$d_L = \frac{1}{a} \int_0^\psi \frac{d\psi}{H(\psi)}. \quad (36)$$

The Hubble derivatives, evaluated as a function of the

⁴ Frequently, cosmologists showed that when different light curve fitters are used, different results with significant discrepancies may be found. For the case of $f(T)$ gravity see for example [34]

redshift z at our time ($z = 0$), give us

$$\frac{dH}{dz} = H_0 (1 + q_0) , \quad (37a)$$

$$\frac{d^2H}{dz^2} = H_0 (j_0 - q_0^2) , \quad (37b)$$

$$(37c)$$

providing a third order Taylor series for the luminosity distance of the form: $\tilde{d}_L = \eta_1 z + \eta_2 z^2 + \eta_3 z^3 + \dots$, where

$$\eta_1 = 1 , \quad (38a)$$

$$\eta_2 = 2 - \frac{3\Omega_{m,0} + \beta \exp(\Omega_{m,0} - 1)}{2} , \quad (38b)$$

$$\eta_3 = \frac{1}{8} \left[3\Omega_{m,0} + \beta \exp(\Omega_{m,0} - 1) \right] \cdot \left[3\Omega_{m,0} - 2 + \beta \exp(\Omega_{m,0} - 1) \right] . \quad (38c)$$

It is useful to stress here that expression (10) represents a general Taylor expansion and may be applied to any cosmological model. The advantage of passing through it is that one directly fits a particular model of interest, weighting the coefficients *directly* with the most recent data. A simple strategy for definitively alleviating the problem of matching data with our model, is to assume *a priori* compatible cosmographic priors. To do so, we employ the theoretical bounds given by (27).

This treatment represents the key to obtain suitable cosmographic intervals, in which the free parameters of our model, i.e. $\Omega_{m,0}$ and β , do not violate the cosmological limits. Our numerical outcomes also need to be compatible with the ones already proposed in the literature and do not have to influence the analyses themselves. In addition, we aim at finding out numerical outcomes over z_{tr} , which can be indirectly derived from the experimental analysis by using (32).

Hence, the Bayesian technique provides the likelihood function:

$$\mathcal{L} \propto \exp(-\chi^2/2) , \quad (39)$$

whose maximum corresponds to the minimum of the χ^2 . We obtain our numerical results performing a test with the free available code ROOT and the additional package BAT [38]. Our analyses are based on two statistical treatments, characterized by different maximum order of parameters. We perform such a procedure in order to provide a hierarchy among all parameters. Firstly, we allow all parameters to freely vary (Fit1), and secondly we fix the mass density parameter through values compatible with the most recent Planck measurements [33] (Fit2).

Our numerical results are summarized in Table I, where we separately report the obtained and the inferred results, showing the limits on the transition redshift itself.

Parameter	Fit1	Fit2
H_0	$69.490^{+0.366}_{-0.379}$	$69.450^{+0.342}_{-0.355}$
α	$1.367^{+0.296}_{-0.252}$	2.067_{---}
β	$-1.147^{+0.696}_{-0.502}$	$0.834^{+0.186}_{-0.168}$
$\Omega_{m,0}$	$0.687^{+0.217}_{-0.185}$	0.274_{---}
z_{tr}	$0.247^{+0.345}_{-0.271}$	$0.643^{+0.034}_{-0.030}$
$ \Delta z_{tr} $	0.385_{---}	0.011_{---}
$P_{T,tr}$	$-0.687^{+0.660}_{-0.641}$	$-1.032^{+0.070}_{-0.063}$

Table I: Table of our experimental and *a posteriori*-derived results. We report the 1σ confidence level errors for our fitting procedure, performed through the Metropolis algorithm. The associated errors on derived quantities have been obtained through the logarithmic rule [1, 39]. To evaluate the transition redshift in the Λ CDM model we considered $\Omega_{m,0} = 0.315$ from the Planck measurements. Finally, H_0 is given in Km/s/Mpc.

The cosmological results show that the first fit (Fit1), in which all coefficients are taken free, does not give conclusive results. In this case, in fact, the mass density is overestimated probably due to the strong multiplicative degeneracy between the coefficients $\Omega_{m,0}$ and β , as one can see from (38). The cosmographic analysis suffers from this kind of degeneracy and shows the same inefficiency in bounding z_{tr} , which seems to significantly departure from the Λ CDM predictions, as shown by looking at the Δz_{tr} . The likelihood contours of this case are shown in Fig. 3.

In the second fit (Fit2), where we fix the matter density parameter to a value compatible with the Planck measurements, namely $\Omega_{m,0} = 0.274$ [33], the results are mostly accurate. This fixing enables to get refined limits even on the other two free coefficients of our model, as can be seen in Fig. 4 (compare with the middle graph of Fig. 3). As a consequence, we obtain more precise bounds on z_{tr} , which becomes perfectly compatible with the constraints predicted by the Λ CDM model, at the $1 - \sigma$ confidence level. Our value however seems to be slightly smaller than theoretical expectations ($z_{tr} = 0.74$ according to [40]).

Hence, we conclude that combined observational tests will represent a landscape to better fix constraints over the involved quantities, showing more accurate limits on the transition predicted by $f(T)$ gravity. Further, this seems to be evident by looking at the contour plots of Figs. 3 and 4, in which a delineated curve corresponds to the second fit. In other words, from these Figures it is clear that the first fit shows higher errors since the contours are larger than the one inferred from the second fit. Nevertheless, in all cases the predicted transition time occurs at $z < 1$, in agreement with the standard theoretical framework.

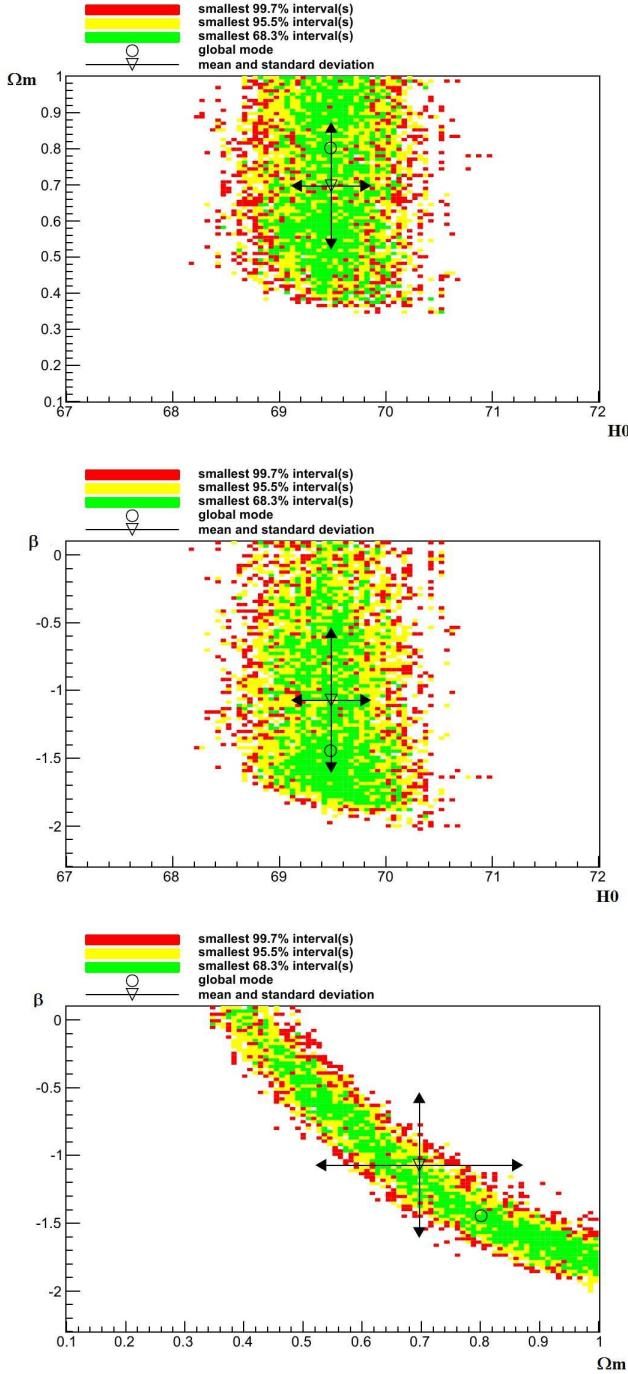


Figure 3: *Contour plots for our observational analyses in the case where all parameters are free to vary (Fit1): $\Omega_{m,0}$ versus H_0 (upper graph), β parameter versus H_0 (middle graph) and β versus $\Omega_{m,0}$ (lower graph).*

VI. CONCLUSIONS AND PERSPECTIVES

In this paper we investigated the transition redshift derived in an effective model inferred from $f(T)$ gravity. In order to do so, we extracted an approximate reconstruction of the $f(T)$ dark-energy term. The ef-

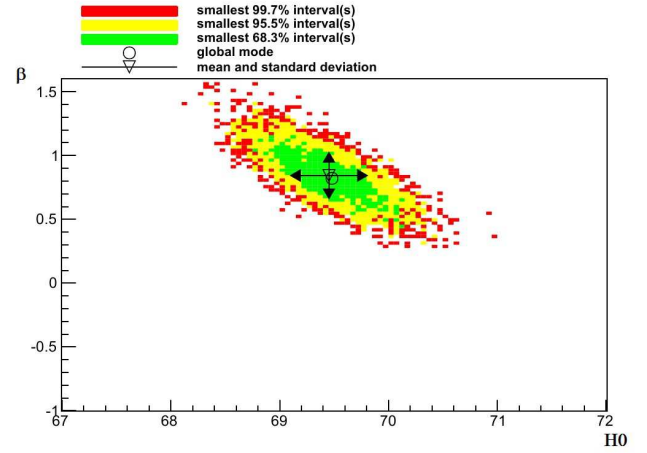


Figure 4: *Contour plots for our observational analyses in the case where only H_0 and β are free to vary, and $\Omega_{m,0}$ is fixed to a value compatible with the Planck measurements, namely $\Omega_{m,0} = 0.274$ (Fit2): β parameter versus H_0 .*

fective dark energy contribution has been obtained by numerically solving the Friedmann equations, employing as initial conditions the numerical outcomes obtained from cosmographic bounds. In this way, we defined a set of numerical constraints on $f(T)$ and its derivatives in a model-independent way, and we were able to fix the evolving dark-energy term through a logarithmic correction.

Our cosmographic model well adapts to the late-time constraints, and it reproduces a cosmological model which smoothly departs from the standard Λ CDM paradigm. The corresponding limits on the free parameters of the model have been obtained by directly fitting the luminosity distance with supernova data, using the most recent union 2.1 compilation. We extracted viable constraints on the free parameters of the scenario, in two distinct fits with different hierarchy between coefficients. We first considered all parameters free to vary and afterwards we fixed the value of the matter density consistently with current Planck results. All predictions provided intervals for the transition redshift which are compatible with present expectations, although the numerical outcomes are slightly smaller than the ones predicted by the standard cosmological model. Departures have been encountered in the case where we leave all parameters free to vary, due to the degeneracy problem between coefficients in the luminosity distance definition. Possible approaches will be devoted to better fix those constraints by means of combined cosmological tests. Moreover, we could mostly investigate the properties of our logarithmic corrections, studying their consequences in the early phases of the universe evolution.

Finally, it would be interesting to extend the above analysis in the case of higher-order torsional cosmology, and in particular in the case where the teleparallel equivalent of the Gauss-Bonnet combination is used in the ac-

tion, as in $f(T, T_G)$ cosmology [41]. The corresponding results could be compared with both Λ CDM cosmology, as well as with the $f(R, \mathcal{G})$ cosmology [42–44]. Such an analysis could provide more information on the possible distinguishability of curvature and torsional gravity using cosmographic methods.

Acknowledgements

S.C. acknowledges INFN Sez. di Napoli (Iniziativa Specifiche CQSKY and TEONGRAV) for financial sup-

port. O.L. wishes to thank Manuel Scinta for the help in the numerical analysis. O.L. is financially supported by the European PONA3 00038F1 KM3NeT (INFN) Project. The research of E.N.S. is implemented within the framework of the Operational Program “Education and Lifelong Learning” (Actions Beneficiary: General Secretariat for Research and Technology), and is co-financed by the European Social Fund (ESF) and the Greek State.

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- [1] S. Capozziello, M. De Laurentis, O. Luongo, A. C. Ruggeri, *Galaxies*, **1**, 216-260, (2013).
 - [2] E. J. Copeland, M. Sami, S. Tsujikawa, *Int. J. Mod. Phys. D*, **15**, 1753-1936, (2006).
 - [3] P. J. E. Peebles, B. Ratra, *Rev. Mod. Phys.*, **75**, 559-606, (2003).
 - [4] S. Capozziello, M. De Laurentis and O. Luongo, arXiv:1411.2822 [gr-qc].
 - [5] K. Bamba, S. Capozziello, S. Nojiri and S.D. Odintsov, *Astrophys. Sp. Sci.*, **342**, 155, (2012).
 - [6] M. Li, X. D. Li, C. Lin, Y. Wang, *Commun. Theor. Phys.*, **51**, 181, (2009); R. Rebolo, et al., *Mon. Not. Roy. Astr. Soc.*, **353**, 747, (2004).
 - [7] E. Linder, *Gen. Rel. Grav.*, **40**, 23, (2008); S. Tsujikawa, arXiv:1004.1493 [astro-ph.CO].
 - [8] S. Weinberg, *Rev. Mod. Phys.*, **61**, 1, (1989).
 - [9] Y. -F. Cai, E. N. Saridakis, M. R. Setare and J. -Q. Xia, *Phys. Rept.* **493**, 1 (2010).
 - [10] S. Capozziello, M. De Laurentis, *Phys. Rept.*, **509**, 167-321, (2011).
 - [11] A. Einstein 1928, *Sitz. Preuss. Akad. Wiss.* p. 217; *ibid* p. 224; A. Unzicker and T. Case, physics/0503046.
 - [12] R. Aldrovandi and J. G. Pereira, *Teleparallel Gravity: An Introduction* (Springer, Dordrecht, 2013); J. W. Maluf, *Annalen Phys.* **525**, 339 (2013).
 - [13] R. Ferraro, F. Fiorini, *Phys. Rev. D*, **75**, 084031, (2007); *Phys. Rev. D*, **78**, 124019, (2008).
 - [14] E. V. Linder, *Phys. Rev. D*, **81**, 127301, (2010) [Erratum-*ibid.* **D**, 82, 109902, (2010)].
 - [15] P. Wu, H. W. Yu, *Phys. Lett. B*, **693**, 415, (2010); S. H. Chen, J. B. Dent, S. Dutta, E. N. Saridakis, *Phys. Rev. D* **83**, 023508 (2011); J. B. Dent, S. Dutta, E. N. Saridakis, *JCAP* **1101**, 009 (2011); K. Bamba, R. Myrzakulov, S. i. Nojiri and S. D. Odintsov, *Phys. Rev. D* **85**, 104036 (2012); G. Otalora, *JCAP* **1307**, 044 (2013); K. Izumi and Y. C. Ong, *JCAP* **1306**, 029 (2013).
 - [16] K. Bamba, C. Q. Geng, C. C. Lee, L. W. Luo, *JCAP*, **1101**, 021, (2011); M. Sharif, S. Rani, *Mod. Phys. Lett. A*, **26**, 1657 (2011); M. R. Setare, M. J. S. Houndjo, *Can. J. Phys.*, **91**, 260-267, (2012); J. Amoros, J. de Haro and S. D. Odintsov, *Phys. Rev. D* **87**, 104037 (2013); G. G. L. Nashed and W. El Hanafy, *Eur. Phys. J. C* **74**, no. 10, 3099 (2014); V. Fayaz, H. Hossienkhani, A. Farmany, M. Amirabadi and N. Azimi, *Astrophys. Space Sci.* **351**, 299 (2014).
 - [17] N. Suzuki, D. Rubin, C. Lidman, G. Aldering, R. Amanullah, et al., *Astrophys. J.*, **746**, 85, (2012).
 - [18] S. Nesseris, S. Basilakos, E. N. Saridakis and L. Perivolaropoulos, *Phys. Rev. D* **88**, 103010 (2013).
 - [19] L. Iorio and E. N. Saridakis, *Mon. Not. Roy. Astron. Soc.* **427**, 1555 (2012).
 - [20] A. Aviles, C. Gruber, O. Luongo, H. Quevedo, *Phys. Rev. D*, **86**, 123516, (2012); A. Aviles, A. Bravetti, S. Capozziello, O. Luongo, *Phys. Rev. D*, **90**, 043531, (2014).
 - [21] S. Capozziello, M. De Laurentis, O. Luongo, *Annal. Phys.*, **526**, 309-317, (2014); C. Catteon, M. Visser, *Class. Quant. Grav.*, **24**, 5985, (2009).
 - [22] A. Aviles, A. Bravetti, S. Capozziello, O. Luongo, *Phys. Rev. D*, **87**, 064025, (2013).
 - [23] C. Clarkson, M. Cortes, B. A. Bassett, *JCAP*, **0708**, 011, (2007).
 - [24] S. Capozziello, O. Luongo, *Cosmographic transition redshift in $f(R)$ gravity*, Proceedings of the conference Quantum Field Theory and Gravity, Tomsk, Russia, (2014), arXiv: 1411.2350; S. Capozziello, O. Farooq, O. Luongo, B. Ratra, *Phys. Rev. D*, **90**, 044016, (2014).
 - [25] A. Aviles, A. Bravetti, S. Capozziello, O. Luongo, *Phys. Rev. D*, **87**, 044012, (2013); B. Bochner, D. Pappas and M. Dong, arXiv:1308.6050 [astro-ph.CO].
 - [26] S. Nesseris, J. Garcia-Bellido, *Phys. Rev. D*, **88**, 063521, (2013).
 - [27] M. Visser, *Gen. Rel. Grav.*, **37**, 1541-1548, (2005).
 - [28] A. R. Neben, M. S. Turner, *ApJ*, **769**, 133, (2013).
 - [29] O. Luongo, *Mod. Phys. Lett. A*, **28**, 1350080, (2013).
 - [30] M. Visser, *Class. Quant. Grav.*, **21**, 2603-2616, (2004).
 - [31] A. G. Riess, et al., *Astrophys. J.*, **607**, 665-687, (2004); R. R. Caldwell, M. Kamionkowski, *JCAP*, **0409**, 009, (2004).
 - [32] O. Farooq, S. Crandall, B. Ratra, *Phys. Lett. B*, **726**, 72-82, (2013).
 - [33] P. A. R. Ade *et al.* [Planck Collaboration], *Astron. Astrophys.* **571**, A16 (2014).
 - [34] G. R. Bengochea, *Phys. Lett. B*, **696**, 5, (2011).
 - [35] R. Amanullah, C. Lidman, D. Rubin, G. Aldering, P. Astier, et al., *Astrophys. J.* **716**, 712-738, (2010).
 - [36] Supernova Cosmology Project Collaboration, M. Kowalski et al., *Astrophys. J.* **686**, 749-778, (2008).
 - [37] M. Goliath *et al.*, *Astron. Astrophys.* **380**, 6, (2008).
 - [38] <http://root.cern.ch/drupal>;
<https://www.mppmu.mpg.de/bat>
 - [39] L. Verde, *Lect. Not. Phys.*, **800**, 147-177, (2010);

- S. Weinberg, *Cosmology*, Oxford Univ. Press, Oxford, (2008).
- [40] O. Farooq, B. Ratra, *Astroph. J.*, **766**, L7, (2013).
 - [41] G. Kofinas and E. N. Saridakis, *Phys. Rev. D* **90**, no. 8, 084044 (2014); G. Kofinas, G. Leon and E. N. Saridakis, *Class. Quant. Grav.* **31**, 175011 (2014); G. Kofinas and E. N. Saridakis, *Phys. Rev. D* **90**, no. 8, 084045 (2014).
 - [42] A. De Felice, J. M. Gerard and T. Suyama, *Phys. Rev. D* **82**, 063526 (2010).
 - [43] M. De Laurentis, *Mod. Phys. Lett. A* **30** (2015) 12, 1550069.
 - [44] M. De Laurentis and A.J. Lopez-Revelles, *Int.J.Geom. Meth. Mod. Phys.* **11** (2014) 1450082 .